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A Bayesian Model for Determining Levels of Student Mastery

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Running head: A BAYESIAN MODEL

Abstract

During the last decade the use of Bayesian statistcial method has become quite prevalent in the educational community. Yet, like most statistical techniques, little has been written concerning the application of these methods to the classroom setting. The purpose of this paper is to help correct such a deficiency in the literature by developing a statistical model using Bayesian concepts which will be of practical use in the classroom. Specifically, the authors develop and discuss a method for using the Bernoulli process in determining a probability of a student reaching a certain level of mastery in a given subject area in which the predetermined prior probabilities and the resultant posterior probabilities are expressions of the probability of a student attaining some level of mastery. Operationally each level of mastery is interpreted as a probability that a student will answer any given question from that topic correctly.

A Bayesian Model for Determining Levels of Student Mastery

The use of Bayesian statistical method has become quite prevalent in the educational community. Yet, like most statistical techniques little has been written concerning the application of these methods to the classroom setting. It is the purpose of this paper to help correct such a deficiency in the literature by developing a statistical model using Bayesian concepts which will be of practical use in the classroom. Specifically, the authors will develop and discuss a method for using the Bernoulli process in determining the probability of a student reaching a certain level of mastery in a given subject area.

To gain a full understanding, and in turn to insure the proper use of any mathematical model, it is necessary to comprehend the underlying concepts and motivations that went into its development. In the case of Bayesian statistics, it would, therefore, seem logical to start with the concepts presented by Bayes in his original paper published posthumously by his friend, Richard Price, in 1963 (Fisher, 1973).

Bayes considered the situation wherein a billiard ball was randomly tossed onto a billiard table and its location used in some predetermined manner to divide the table into two rectangles which were then conveniently labeled L and R. An

experiment then took place in which the ball was again randomly thrown onto the table n times. If the ball landed in region L, the trial was considered a success; if the ball landed in region R, a failure was recorded. At this point Bayes asked the following question: If neither the area of L nor of R were known, could the number of subsequent successes and failures be used to determine the probability of some previously assumed division of the table?

If we assumed only a finite number of values for the area of L, then Bayes demonstrated that

$$P(A = a_i \mid Y = y) = \frac{P(Y = y \mid A = a_i)\, P(A = a_i)}{P(Y = y \mid A = a_i)\, P(A = a_i)}$$

Here the $P(A = a_i \mid Y = y)$ is simply the probability that the area of L is equal to a_i given that there were y successes in the experiment. The $P(Y = y \mid A = a_i)$ is the probability that there were y successes given an area of L equal to a_i. Finally, the $P(A = a_i)$ is the original probability that the area of L equals a_i. It must be mentioned that A and Y are random variables with A representing the area of L and Y representing the number of successes in the experiment.

An intuitive understanding of the above formula, which is

one form of Bayes' theorem, can be gained by considering the following example. Let A be a fraction of the total area of L and have the following values: $A = \frac{1}{4}$, $A = \frac{2}{4}$, $A = \frac{3}{4}$, $A = \frac{4}{4}$. Let $Y = 3$ and let $n = 10$ be the number of trials in the experiment. The problem is to determine the $P(A = \frac{x}{4} \mid Y = 3)$ for all integer values of x between 1 and 4 inclusive.

What are called prior probabilities must first be determined. These are nothing more than the probabilities of A having a certain value before the experiment is done. In this example, it will be assumed each value of A is equally likely and hence $P(A = \frac{1}{4}) = P(A = \frac{2}{4}) = P(A = \frac{3}{4}) = P(A = \frac{4}{4}) = .25$. This assumption is considered reasonable only if we know the procedure whereby L was determined. Here we will assume we know this information.

Likelihoods, that is the probability that $Y = 3$ given a value of A, must next be determined. Here the binomial distribution works nicely (Winkler, 1972), and the likelihoods are as follows:

$$P(Y = 3 \mid A = \tfrac{1}{4}) = \binom{10}{3}(.25)^3(.75)^7 = .2503$$

$$P(Y = 3 \mid A = \tfrac{2}{4}) = \binom{10}{3}(.5)^3(.5)^7 = .1172$$

$$P(Y = 3 \mid A = \tfrac{3}{4}) = \binom{10}{3}(.75)^3(.25)^7 = .0031$$

$$P(Y = 3 \mid A = \tfrac{4}{4}) = \binom{10}{3}(1)^3(0)^7 = 0$$

Notice that the sum of these likelihoods does not equal one, since we are dealing with four different binomial distributions.

Next, the probabilities of compound events $(Y = 3 \mid A = \frac{x}{4})$ and $(A = \frac{x}{4})$ can be seen to be the products $P(Y = 3 \mid A = \frac{x}{4})P(A = \frac{x}{4})$. Inserting specific values for A, we have

$$P(Y = 3 \mid A = \tfrac{1}{4})P(A = \tfrac{1}{4}) = .0625,$$

$$P(Y = 3 \mid A = \tfrac{2}{4})P(A = \tfrac{2}{4}) = .0293,$$

$$P(Y = 3 \mid A = \tfrac{3}{4})P(A = \tfrac{3}{4}) = .0008,$$

and $$P(Y = 3 \mid A = \tfrac{4}{4})P(A = \tfrac{4}{4}) = 0.$$

From these figures it would be expected that for every 10,000 times the experiment was run, 625 results would occur wherein $Y = 3$ and $A = \frac{1}{4}$. Likewise, it would be expected that $Y = 3$ 625 + 293 + 8 + 0 = 926 times every 10,000 experiments. Hence, since in this example it is already known that $Y = 3$,

$$P(A = \tfrac{1}{4} \mid Y = 3) = \frac{625}{926} = .6749,$$

$$P(A = \tfrac{2}{4} \mid Y = 3) = \frac{293}{926} = .3164,$$

$$P(A = \tfrac{3}{4} \mid Y = 3) = \frac{8}{926} = .0086,$$

and $$P(A = \tfrac{4}{4} \mid Y = 3) = \frac{0}{926} = 0.$$

The above argument is summarized in Table 1. The first column lists all the possible values for A, and the second and third columns give the prior probabilities and the likelihoods with the fourth column being the product of these two. The fifth column is obtained by summing the fourth column and then dividing each entry in the fourth column by that sum. It is easily seen that this last procedure is equivalent to the frequency approach used earlier.

Insert Table 1 about here

A Bayesian Model

Teachers are continuously faced with the problem of when to move on to new material. Whether the new material demands an understanding of the topics just covered or whether the teacher is just interested in a certain level of mastery, the question remains an important one. In making this decision, a large amount of subjective as well as objective data is considered. Thorndike (1980), Hambleton and others (1976), and Marshall (1968) discuss the usefulness of Bayesian statistical methods in combining prior data of different types concerning a student's mastery of subject matter with subsequent objective

testing. This section will present a Bayesian technique to help determine student mastery which is practicable for classroom use.

The specific Bayesian method presented here assumes that the objective test used to measure a student's mastery of the given subject material can be considered a Bernoulli process. The predetermined prior probabilities and the resultant posterior probabilities are expressions of the probability of a student attaining some level of mastery. Operationally each level of mastery is interpreted as a probability that a student will answer any given question from that topic correctly. Thus, a 65% level of mastery means that a student has a 65% chance of answering a relevant question correctly. This technique of measuring a student's mastery of material is different from the cutoff score technique used by most criterion-referenced testing advocates. Yet, the author feels that this interpretation is more realistic, agreeing with Glass (1978) on the impossibility of making statements of certainty concerning a student's comprehension of an idea or concept. With these opening remarks concluded, the discussion can now turn to the specific Bayesian method of this paper.

In general, this technique involves four steps. In the first the teacher develops an objective test within the confines of the Bernoulli process. The teacher then determines the prior

probabilities by using all relevant data to arrive at the probability the student will fall into each level of mastery. The test is then administered and using the binomial distribution, likelihoods are calculated. Finally, using Bayes' theorem, the posterior probability for each level of mastery is determined and these are used to decide on the next class activity.

In developing any test as a Bernoulli process, several assumptions must be obeyed. The Bernoulli process first requires that each trial, or test question, must produce only one of two possible outcomes. Also subsumed within the Bernoulli process is the stationarity and independence of each trial. Stationarity means that the probability a student will answer each question correctly remains constant. Independence simply requires that the answer to any one question not affect the answer to any other. It must be remembered that, although it may not be possible to hold to these assumptions exactly, the more severely they are violated, the more suspect is the data generated by the model.

The determination of prior probabilities is both the strength and weakness of any Bayesian technique. In this situation it is being assumed that the information available to the teacher prior to testing is as reliable as the information provided by the test itself. This does not seem to be an

unreasonable assumption but in the end the teacher must make this decision. If the teacher feels that prior data is not as reliable as the subsequent test data, then the test data alone should be used.

To determine the likelihoods, the binomial distribution is used in the following form:

$$P(r \mid p) = \frac{n!}{r!(n - r)!}(p)^{r}(1 - p)^{n - r}.$$

In this distribution n is the number of test questions, r the number of correct answers on the test, and p is the probability the student will answer any given question correctly.

At this point a table is constructed in order to produce the posterior probabilities. This final step will be more easily demonstrated in the following example:

An Example

Consider a situation where a seventh grade teacher has just finished working with one of his students on how to find least common multiples. Before he stops seeing this student, he would like to arrive at some indication of the student's level of mastery of the material. The teacher feels that while working with the student he has gained a fair estimation of his ability; yet, he would still like to obtain some type of objective

measure. He therefore decides to use Bayes' theorem to determine the probability that his student has attained any given level of mastery.

The teacher first decides to measure the student on ten distinct levels of mastery. These are 5%, 15%, 25%, 35%, 45%, 55%, 65%, 75%, 85%, and 95%. He then assigns prior probabilities as follows:

Levels of Mastery

5%	15%	25%	35%	45%	55%	65%	75%	85%	95%
.00	.00	.05	.05	.10	.20	.25	.20	.10	.05

Prior Probabilities

The teacher then administers an objective test on which the student answers five out of ten questions correctly. From this information the following likelihoods were indicated:

$$P(5 \mid .25) = \binom{10}{5} (.25)^5(.75)^5 = .0584$$

$$P(5 \mid .35) = \binom{10}{5} (.35)^5(.65)^5 = .1536$$

$$P(5 \mid .45) = \binom{10}{5} (.45)^5(.65)^5 = .2340$$

$$P(5 \mid .55) = \binom{10}{5} (.55)^5(.45)^5 = .2340$$

$$P(5 \mid .65) = \binom{10}{5}(.65)^5(.35)^5 = .1536$$

$$P(5 \mid .75) = \binom{10}{5}(.75)^5(.25)^5 = .0584$$

$$P(5 \mid .85) = \binom{10}{5}(.85)^5(.15)^5 = .0085$$

$$P(5 \mid .95) = \binom{10}{5}(.95)^5(.05)^5 = .0001$$

The teacher then summarizes all of the data in tabular form. After looking at this table the teacher decides to work a little more with his student.

Insert Table 2 about here

Discussion

In 1968 Marshall began his paper with the statement that "The Bayesian approach to statistics has been both damned and praised by behavioral researchers" (p. 71). Little has changed in the past couple of decades. At the heart of the argument is the determination of prior probabilities. In situations where these probabilities are easily determined, there is little agrument over the use of Bayes' theorem. This view is evidenced by Mallows (1969) in the following:

> From reading the literature one might suppose that a major part of the effort in executing a typical Bayesian analysis is devoted to computing likelihoods and hence posterior probabilities. I would regard this activity as an exercise in probability theory. . . . The mere use of Bayes theorem is not a defining event! Even frequentists use conditional probability. (p. 72)

But when Bayes' theorem is used to determine individual parameters in objective testing, the bias produced by certain methods of determining priors is quite an obstacle to its use in this field (Samejimo, 1980).

There is a strong reason why this problem in objective testing does not extend to the classroom use of Bayes' theorem. Any good teacher possesses information on a student's abilities which is as relevant and reliable as any objective test. The proper use of an objective test in the classroom is to lend objective support to a teacher's prior belief concerning a student's abilities. It is an evaluation of progress which should have already been observed. It therefore seems logical to use Bayes' theorem to combine both the objective and subjective data available to the teacher.

In this paper the author suggests one way to use information obtained from an application of Bayes' theorem to the problem of determining the level of student mastery. It was suggested that

this information be used to decide when to move on to new material. There may be other effective ways to use Bayesian statistics, but there is one way the authors feel it should not be used. In a 1981 paper, Powers suggested that a Bayesian method, similar to the one presented here, be used to assign course grades. It appears to these writers that to review certain material, even though a student scored 80% on an objective test, because the teacher had predicted a low mastery state for that individual, is a reasonable action. It does not appear reasonable to assign a student a C for the course, even though he earned a B, because the teacher expected him to make a D.

In closing, it is hoped that this paper will provide a stimulus for the development of other relevant Bayesian techniques for the classroom. Although other methods derived from Bayes' theorem are more complex, the increased presence of computers in the school should provide many interesting developments.

References

Fisher, R. A. (1973). Statistical methods and scientific inference. New York: Hafner Press.

Glass, G. V. (1978). Standards and criteria. Journal of Educational Measurement, 15, 237-281.

Hambleton, R. K., Hutten, L. R., & Swaminathan, H. A. (1976). Comparison of several methods for assessing student mastery in objective-based instructional programs. Journal of Educational Measurement, 45, 57-64.

Mallows, C. L. (1970). Some comments on Bayesian methods. In D. L. Meyer and R. O. Collier, Jr. (Eds.), Bayesian Statistics. (pp. 71-84). Itasca, IL: F. E. Peacock Publishers.

Marshall, J. C. (1968). Bayesian inference. Journal of Experimental Education, 37, 71-75.

Powers, J. E. (1981). A Bayesian method for assigning course grades. Lenox, MA: Paper presented at the annual meeting of the New England Educational Research Organization. (ERIC Document Reproduction Service No. ED 013 371)

Samejimo, F. (1980). Is Bayesian estimation proper for estimating the individual's ability? Research Report 80-3. (ERIC Document Reproduction Service No. ED 193 305)

Thorndike, R. L. (1980). The application of Bayesian thinking to educational measurement problems. Sydney, Austrailia:

Invitational address to the Victorian Institute of Educational Research. (ERIC Document Reproduction Service No. ED 194 553)

Winkler, R. L. (1972). Introduction to Bayesian inference and decision. New York: Holt, Rinehart, and Winston.

Table 1

The Probabilities of Compound Events When Y = 3 and A = $\frac{x}{4}$

A	Prior Probability	Likelihoods	(Prior Probabilities) X (Likelihoods)	Posterior Probabilities
$\frac{1}{4}$	.25	.2503	.0625	.6749
$\frac{2}{4}$	.25	.1172	.0293	.3164
$\frac{3}{4}$	.25	.0031	.0008	.0087
$\frac{4}{4}$	.25	.0000	.0000	.0000
	1.00		0.0926	1.0000

Note. Y equals successes in the experiment and A equals the area of L.

Table 2

Probability of Answering Five Out of Ten Questions Correctly Based on Specified Levels of Mastery and Prior Probabilities

Level of Mastery	Prior Probability	Likelihood	(Prior Probability) X (Likeli-ood)	Posterior Probability
95%	.05	.0001	.0000	.0000
85%	.10	.0085	.0009	.0067
75%	.20	.0584	.0117	.0888
65%	.25	.1536	.0384	.2913
55%	.20	.2340	.0468	.3551
45%	.10	.2340	.0234	.1775
35%	.05	.1536	.0077	.0584
25%	.05	.0584	.0029	.0220
15%	.00	.0000	.0000	.0000
5%	.00	.0000	.0000	.0000
	1.00		.1318	1.000

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